

ALGEBRAS WHICH ARE NEARLY FINITE DIMENSIONAL AND THEIR IDENTITIES

BY

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ABSTRACT

Suppose that all the nonzero one-sided or two-sided ideals of an algebra have finite codimension. To what extent must the algebra be p.i. or primitive?

Introduction

Finite-dimensional algebras are regarded as all but trivial examples of polynomial identity algebras. In this note, we examine algebras which just miss being finite dimensional in the sense that many of their nonzero one-sided or two-sided ideals have finite codimension. The vague goal is to prove that such algebras still satisfy a polynomial identity. Short of that, we wish to establish some sort of “alternative” for those algebras which are also semiprimitive, e.g., they must be either p.i. or primitive.

We apply this philosophy in providing a new proof of part of the Small–Warfield Theorem [SW] describing prime affine algebras with Gelfand–Kirillov dimension

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one. An “alternative” for affine algebras over an uncountable scalar field is also presented in Theorem 2.2.

This exploration arose from stimulating discussions the first author had with Ed Letzter, for which he is very grateful. The significance of the main lemma became clear when the two authors played with it at MSRI, under the auspices of the special year for noncommutative algebra 1999–2000.

1. Creating polynomial identities

We shall describe how polynomial identities arise for algebras with sufficiently many left ideals of finite codimension. Let R be a k -algebra where k is a field. Denote by $T_n(R)$ the set of all specializations in R by all polynomial identities of $n \times n$ matrices over k . Obviously, $T_n(R)$ is an ideal of R and $R/T_n(R)$ is a p.i. algebra.

LEMMA 1.1: *Assume R is a k -algebra and n is a positive integer. If $a \in R$ is such that*

$$\dim_k R/Ra = n,$$

then $T_n(R) \subseteq \bigcap_d Ra^d$.

Proof: Pull back a basis for R/Ra to a set $v_1, \dots, v_n$ in R . We may assume $v_1 = 1$. A routine induction shows that if $s \in R$ and d is a natural number then we may write

$$s \equiv v_1 p_1(a) + \dots + v_n p_n(a) \pmod{Ra^{d+1}}$$

where $p_j(a) \in k[a]$ has formal degree at most d as a polynomial in a . For the time being, we fix d . First apply the formula to $s = rv_j$ for an arbitrary $r \in R$, thereby constructing the essence of the left regular representation modulo Ra^{d+1} . Then we are writing

$$rv_j = \sum_i v_i r_{ij} + f_j(r)a^{d+1}$$

with $r_{ij} \in k[a]$ and $f_j(r) \in R$. If $t \in R$ then

$$\begin{aligned} t(rv_j) &= \sum_i tv_i r_{ij} + tf_j(r)a^{d+1} \\ &= \sum_{m,i} v_m t_{mi} r_{ij} + \sum_i f_i(t)a^{d+1} r_{ij} + tf_j(r)a^{d+1}. \end{aligned}$$

Notice that the middle term lies in Ra^{d+1} because $r_{ij} \in k[a]$ and polynomials in the symbol a commute.

We reinterpret the last equation using matrix notation: write $[r] = (r_{ij})$, etc. All of the matrices which appear will be $n \times n$ matrices with entries in $k[a]$. (We stress that n is independent of d .) We have

$$(tr) \cdot (v_1, \dots, v_n) \equiv (v_1, \dots, v_n)[t][r] \pmod{Ra^{d+1}}.$$

Here the dot refers to “scalar multiplication” for R acting on the R -module R^n and on the right-hand side of the congruence we have the product of a row vector with the product of two matrices.

The congruence can be extended to arbitrarily long products. In other words, if F is a noncommutative polynomial in ℓ variables and $r^{(1)}, \dots, r^{(\ell)} \in R$ then

$$F(r^{(1)}, \dots, r^{(\ell)}) \cdot (v_1, \dots, v_n) \equiv (v_1, \dots, v_n)F([r^{(1)}], \dots, [r^{(\ell)}])$$

modulo Ra^{d+1} . In particular, if F is a polynomial identity for $n \times n$ matrices (i.e., for $n \times n$ matrices over $k[a]$) then

$$F(r^{(1)}, \dots, r^{(\ell)}) \cdot (v_1, \dots, v_n) \equiv 0 \pmod{Ra^{d+1}}$$

for all specializations.

Since $v_1 = 1$, attention to the first coordinate tells us that

$$F(r^{(1)}, \dots, r^{(\ell)}) \in Ra^{d+1}.$$

Equivalently,

$$T_n(R) \subseteq Ra^{d+1}.$$

Let d vary. ■

We shall take advantage of the conclusion of Lemma 1.1 by means of the next elementary lemma.

LEMMA 1.2: *Assume R is a k -algebra. If $a \in R$ is regular and $\bigcap_d Ra^d$ has finite codimension in R then a is invertible.*

Proof: The chain

$$Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \dots$$

must eventually be stationary. Thus $a^t \in Ra^{t+1}$ for some positive integer t . It follows that a has a left inverse. Regularity of a implies that this left inverse is also a right inverse. ■

The Small–Warfield Theorem asserts that a prime affine algebra of GK-dimension one satisfies a polynomial identity. The first half of their argument is devoted to proving that the hypotheses on the algebra force it to be Goldie. We offer an alternative approach to the completion of the argument.

LEMMA 1.3: Assume R is a prime Goldie k -algebra of GK-dimension one. Then either R is simple artinian or it satisfies all polynomial identities of $n \times n$ matrices for some n .

Proof: If every regular element in R is invertible then Goldie's Theorem states that R is simple artinian. So we may suppose that R contains some regular element a which is not invertible.

We shall invoke the following observation twice: if L is a left ideal of R which contains a regular element then the GK-dimension of R/L is zero, i.e., $\dim_k R/L < \infty$. (See [McR], 8.35.)

First take $L = Ra$. According to Lemma 1.1,

$$T_n(R) \subseteq \bigcap_d Ra^d.$$

We are clearly done when $T_n(R) = 0$. Otherwise, $T_n(R)$ is a nonzero two-sided ideal. As such it must contain a regular element because R is prime Goldie. This time use $L = \bigcap Ra^d$ to conclude that the intersection has finite codimension in R . Lemma 1.2 now implies that a is invertible, a contradiction. ■

Observe that if k is an uncountable, algebraically closed field then the punch-line of the Small–Warfield Theorem follows immediately from the lemma. However, we can always reduce to this case. Let K be an uncountable, algebraically closed extension field of k . Use Zorn's Lemma to find an ideal I of $K \otimes_k R$ maximal with respect to $I \cap R = 0$. The primality of R forces I to be a prime ideal of $K \otimes R$.

It is easy to see that if R has GK-dimension one as a k -algebra then $K \otimes R$ has GK-dimension one as a K -algebra. All the more so, $(K \otimes R)/I$ has GK-dimension one. The first part of Small–Warfield (which we are assuming to be true) plus Lemma 3.1 say that $(K \otimes R)/I$ satisfies a standard identity. Since R imbeds in $(K \otimes R)/I$, it, too, satisfies a standard identity.

We turn next to the cheapest way to guarantee the presence of many left ideals with finite codimension.

PROPOSITION 1.1: Assume that R is a k -algebra such that every nonzero left ideal has finite codimension in R . Then either R is finite dimensional over k , a division algebra, or a domain module-finite over its center.

Proof: We will assume that R is not finite dimensional over k . Then R must be an integral domain. Indeed, if $a \neq 0$ is in R then R/Ra is finite dimensional;

thus Ra is infinite dimensional over k . But

$$Ra \simeq R/\text{lann}_R(a).$$

Hence the left annihilator of a is zero.

Suppose, in addition, that R is not a division algebra. Then R has a regular element which is not invertible. Lemmas 1.1 and 1.2 imply that $T_n(R) = 0$. Therefore R satisfies a polynomial identity.

We claim that the center $\mathcal{Z}$ of R is noetherian. If $0 \neq w \in \mathcal{Z}$ then $wR \cap \mathcal{Z} = w\mathcal{Z}$ because w is regular. It follows that all nonzero ideals of $\mathcal{Z}$ have finite codimension in $\mathcal{Z}$. In particular, $\mathcal{Z}$ is noetherian. The proposition is completed by invoking Formanek's module finiteness theorem (see [McR], 13.6.11). ■

Note that if k is algebraically closed and R is an affine k -algebra then Tsen's Theorem ([G]) allows us to replace the third alternative with " R is a commutative affine domain of Krull dimension one".

It may be argued that the proposition is a pleasing characterization with the wrong hypothesis. There is evidence from the study of pathological infinite groups that it is more natural to restrict finite codimension to two-sided ideals only.

Definition: An infinite dimensional k -algebra R is **just infinite dimensional** provided that every nonzero two-sided ideal has finite codimension in R .

2. Just infinite dimensional algebras

There is a very small literature on just infinite dimensional algebras: [S], [PT], [V]. We begin our advertisement for this subject with a particularly simple proof of the main result in [F].

LEMMA 2.1 (Small [PT]): *Let R be an affine k -algebra which is not finite dimensional. Then R has a prime ideal I such that R/I is just infinite dimensional.*

As a consequence, just infinite dimensional affine algebras are prime.

THEOREM 2.1 ([F]): *An affine polynomial identity algebra for which every finite-dimensional module is semisimple must be a finite-dimensional algebra.*

Proof: If there is an infinite-dimensional counterexample then there is an example R which is just infinite dimensional by the lemma. We claim that the center of R is a field. Certainly every nonzero element of the center is regular, as argued in Proposition 1.1. If z is such an element which is not invertible then Rz^2 is a proper nonzero ideal of R . Hence R/Rz^2 is a finite-dimensional semisimple

k -algebra. But the image of z lies in its radical! It follows that $z \in Rz^2$, so z is invertible after all.

According to Rowen's Theorem, every nonzero ideal of a prime p.i. algebra nontrivially meets the center. Thus R is simple. By Kaplansky's Theorem, R is finite dimensional over its center. The Artin–Tate Lemma ([McR], 13.9.10) implies that the center is affine over k . Finally, the classical Nullstellensatz says that the center is a finite field extension of k . ■

The examples in [S] and [V] strongly suggest an “alternative” theorem for affine semiprimitive just infinite dimensional algebras: either they are primitive or they satisfy a p.i. We offer a proof valid over large scalar fields.

THEOREM 2.2: *Assume that k is an uncountable field. If R is an affine, semiprimitive, just infinite dimensional k -algebra then either R is (left) primitive or R satisfies a polynomial identity.*

Proof: Suppose that neither conclusion holds. Then each primitive ideal of R has finite codimension. We first show that there are only countably many primitive ideals. The point is that for each positive integer n there are only finitely many primitive ideals with codimension n ; otherwise, their intersection will be zero, whence R satisfies a polynomial identity of degree $n^2 + 1$.

Fix $a \in R$. If P is a primitive ideal of R then R/P is a finite-dimensional simple algebra. Hence the image of $a - \lambda \cdot 1$ in R/P is invertible for all but finitely many $\lambda \in k$. Using the previous paragraph, we see that there is a countable subset $C \subseteq k$ such that $a - \lambda \cdot 1$ is invertible modulo every primitive ideal whenever $\lambda \notin C$.

We shall prove that if $\lambda \notin C$ then $a - \lambda \cdot 1$ itself has a left inverse in R . Suppose not. Then $R(a - \lambda \cdot 1)$ lies in some maximal left ideal $\mathcal{M}$. If P is the annihilator of $R/\mathcal{M}$ then P is primitive. Consequently, the image of $a - \lambda \cdot 1$ in R/P lies in a maximal ideal of R/P , contradicting its invertibility modulo P .

We have proved that $a - \lambda \cdot 1$ has a left inverse for uncountably many scalars λ . Amitsur's linear independence trick implies that a is algebraic over k . Summarizing, R is an algebraic algebra over k . Since R is semiprimitive, it has no nonzero nil left ideals. Hence every nonzero left ideal of R contains a nontrivial idempotent.

We reach the final contradiction that R is primitive or p.i. via a Baire Category argument due to Jacobson. The relevant version is the elegant formulation in [FS]. Assume that R is any prime ring with the property that every nonzero left ideal has a nontrivial idempotent. Their theorem states that if there exists a

countable collection of nonzero two-sided ideals such that every nonzero ideal of R contains an ideal in the collection, then R is primitive. In our case, the countable sequence of distinguished ideals are the $T_n(R)$. If one of them is zero then R satisfies a p.i. If none are zero, then this collection is suitable because each nonzero ideal of R has finite codimension. ■

A similar argument (which does not require Baire Category) has been used by the second author to prove (unpublished) that if R is a prime, affine, noetherian, $\mathbf{C}$ -algebra of Krull dimension 1 then either R is primitive or R satisfies a polynomial identity.

The hypothesis of semiprimitivity for Theorem 2.2 is necessary. Let S be a Golod–Shafarevich example, a finitely generated nil algebra which is not nilpotent. Zorn’s Lemma produces an ideal J maximal with respect to S/J not being nilpotent because the inclusion of S^m in an ideal can be determined by testing products of m generators. It is immediate that S/J with 1 adjoined is just infinite dimensional. Finitely generated nil algebras which satisfy a polynomial identity are nilpotent, so the algebra we have constructed satisfies no identity.

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